Recursively Defined Functions

- We can use the following method to define a function with the **natural numbers** as its domain:

- Base case: Specify the value of the function at zero.
- Recursion: Give a rule for finding its value at any integer from its values at smaller integers.

- Such a definition is called **recursive** or **inductive definition**.

Recursively Defined Functions

- **Example:**

- $f(0) = 3$
- $f(n + 1) = 2f(n) + 3$

- $f(0) = 3$
- $f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$
- $f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$
- $f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$
- $f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$

Recursively Defined Functions

- **How can we recursively define the factorial function $f(n) = n!$?**

- $f(0) = 1$
- $f(n + 1) = (n + 1)f(n)$

- $f(0) = 1$
- $f(1) = 1f(0) = 1 \cdot 1 = 1$
- $f(2) = 2f(1) = 2 \cdot 1 = 2$
- $f(3) = 3f(2) = 3 \cdot 2 = 6$
- $f(4) = 4f(3) = 4 \cdot 6 = 24$

Recursively Defined Functions

- **A famous example: The Fibonacci numbers**
- $f(0) = 0, f(1) = 1$
- $f(n) = f(n – 1) + f(n - 2)$
- $f(0) = 0$
- $f(1) = 1$
- $f(2) = f(1) + f(0) = 1 + 0 = 1$
- $f(3) = f(2) + f(1) = 1 + 1 = 2$
- $f(4) = f(3) + f(2) = 2 + 1 = 3$
- $f(5) = f(4) + f(3) = 3 + 2 = 5$
- $f(6) = f(5) + f(4) = 5 + 3 = 8$

Recursively Defined Sets

- If we want to recursively define a set, we need to provide two things:

- an **initial set** of elements,
- **rules** for the construction of **additional** elements from elements in the set.

- **Example:** Let S be recursively defined by:
- $3 \in S$
- $(x + y) \in S$ if $(x \in S)$ and $(y \in S)$

- S is the set of positive integers divisible by 3.

Recursively Defined Sets

- **Proof:**

- Let A be the set of all positive integers divisible by 3.

- To show that A = S, we must show that
$A \subseteq S$ and $S \subseteq A$.

- Part I: To prove that $A \subseteq S$, we must show that every positive integer divisible by 3 is in S.

- We will use mathematical induction to show this.

Recursively Defined Sets

- Let P(n) be the statement "3n belongs to S".

- **Basis step:** P(1) is true, because 3 is in S.

- **Inductive step:** To show:

If P(n) is true, then P(n + 1) is true.

- Assume 3n is in S. Since 3n is in S and 3 is in S, it follows from the recursive definition of S that
$3n + 3 = 3(n + 1)$ is also in S.

- **Conclusion of Part I:** $A \subseteq S$.

Recursively Defined Sets

- Part II: To show: $S \subseteq A$.

- **Basis step:** To show:
All initial elements of S are in A. 3 is in A. True.

- **Inductive step:** To show:
If x and y in S are in A, then $(x + y)$ is in A .

- Since x and y are both in A, it follows that $3 \mid x$ and $3 \mid y$.
From Theorem I, Section 2.3, it follows that $3 \mid (x + y)$.

- **Conclusion of Part II:** $S \subseteq A$.
- **Overall conclusion:** $A = S$.

Recursively Defined Sets

- **Another example:**

- The well-formed formulae of variables, numerals and operators from {+, -, *, /, ^} are defined by:

- x is a well-formed formula if x is a numeral or variable.

- (f + g), (f – g), (f * g), (f / g), (f ^ g) are well-formed formulae if f and g are.

Recursively Defined Sets

- With this definition, we can construct formulae such as:

- (x – y)
- ((z / 3) – y)
- ((z / 3) – (6 + 5))
- ((z / (2 * 4)) – (6 + 5))

Recursive Algorithms

- An algorithm is called **recursive** if it solves a problem by reducing it to an instance of the same problem with smaller input.

- **Example I:** Recursive Euclidean Algorithm

- **procedure** gcd(a, b: nonnegative integers with a < b)
- **if** a = 0 **then** gcd(a, b) := b
- **else** gcd(a, b) := gcd(b **mod** a, a)

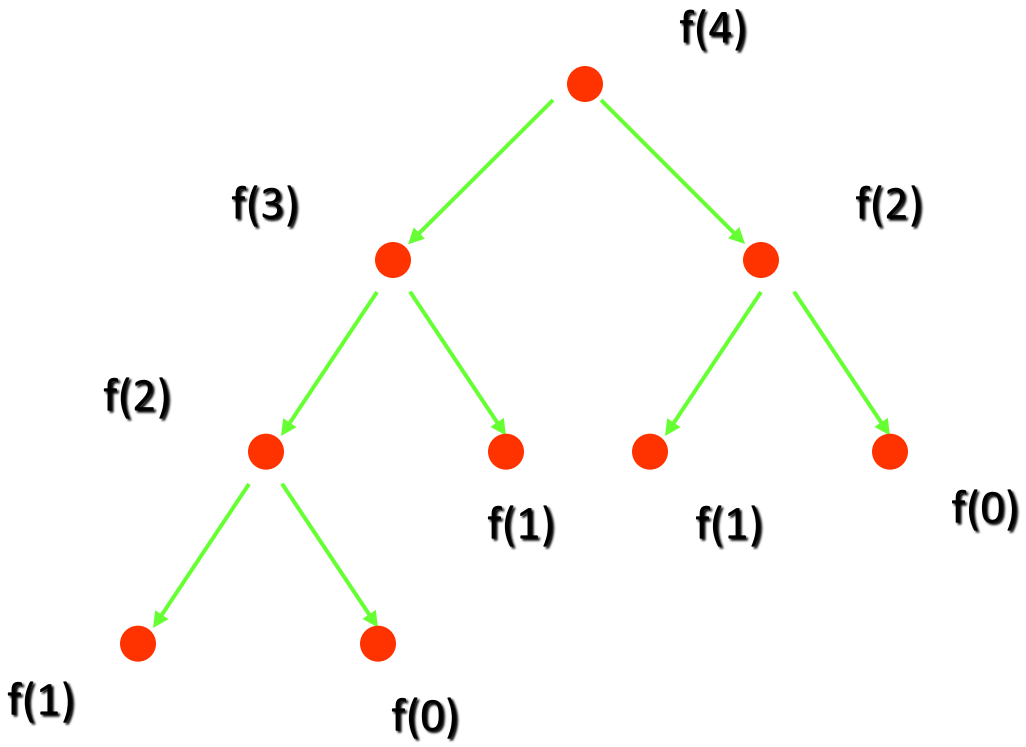
Recursive Algorithms

- **Example II:** Recursive Fibonacci Algorithm

- **procedure** fibo(n: nonnegative integer)
- **if** n = 0 **then** fibo(0) := 0
- **else if** n = 1 **then** fibo(1) := 1
- **else** fibo(n) := fibo(n – 1) + fibo(n – 2)

Recursive Algorithms

- Recursive Fibonacci Evaluation:



Recursive Algorithms

```
•procedure iterative_fibo(n: nonnegative integer)
•if n = 0 then y := 0
•else
•begin
•          x := 0
•          y := 1
•          for i := 1 to n-1
•          begin
•                    z := x + y
•                    x : = y
•                    y := z
•          end
•end   {y is the n-th Fibonacci number}
```

Recursive Algorithms

- For every recursive algorithm, there is an equivalent iterative algorithm.

- Recursive algorithms are often shorter, more elegant, and easier to understand than their iterative counterparts.

- However, iterative algorithms are usually more efficient in their use of space and time.